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Some relations on Miyazawa's virtual knot invariant ☆

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Abstract

Y. Miyazawa defined a polynomial invariant for a virtual link by using magnetic graph diagrams, which is related with the Jones–Kauffman polynomial. In this paper we show some relations of this polynomial for a virtual skein triple.

MSC: primary 57M25; secondary 57M27

Keywords: Virtual knot theory

1. Introduction

A *virtual link diagram* is a link diagram in $\mathbb{R}^2$ possibly with some encircled crossings without over/under information, called *virtual crossings*. A *virtual link* [7] is the equivalence class of such a link diagram by generalized Reidemeister moves illustrated in Fig. 1.

In [7], Kauffman defined a polynomial invariant $f_L(A) \in \mathbf{Z}[A^2, A^{-2}]$ for a virtual link L, which we call the Jones–Kauffman polynomial. For a classical link L, it is equal to the Jones polynomial $V_L(t)$ after substituting $\sqrt{t}$ for A^2. The Miyazawa polynomial is a 2-variable invariant for a virtual link whose value is in $\mathbf{Z}[A, A^{-1}, h]$. (See [4] or Section 2 for the definition.) It is equal to the Jones–Kauffman polynomial $f_L(A)$ after substituting 1 for h. For a virtual link diagram D or its equivalence class $L = [D]$, we denote the Miyazawa polynomial by $R_D(A, h)$ or $R_L(A, h)$.

Let D_+ be a virtual link diagram of which a real crossing p_+ is positive and D_- (or D_v) the virtual link diagram obtained from D_+ by replacing p_+ with a negative crossing (or a virtual crossing) illustrated in Fig. 2. Let D_0 be the virtual link diagram obtained from D_+ by smoothing at a positive crossing p_+ such that the orientation of D_0 is induced from that of D_+ except a regular neighborhood $N(p_+)$ of p_+ illustrated in Fig. 2. We call (D_+, D_-, D_0) a *skein triple*, and we call (D_+, D_-, D_v) a *virtual skein triple* with respect to a positive crossing p_+ of D_+.

Let (D_+, D_-, D_0) be a skein triple of virtual link diagrams. In [7] it is proved that

$$A^4 f_{D_+}(A) - A^{-4} f_{D_-}(A) + \left(A^2 - A^{-2}\right) f_{D_0}(A) = 0.$$

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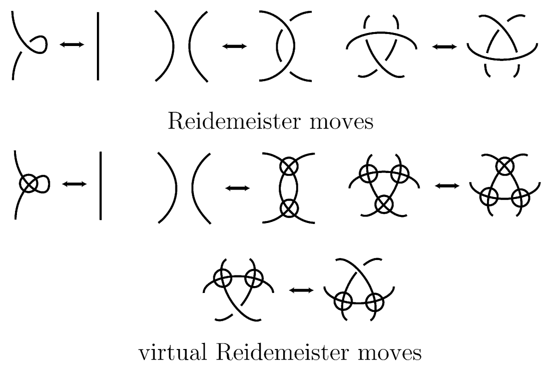

Fig. 1.

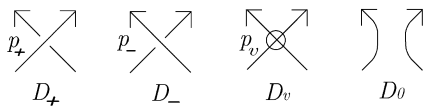

Fig. 2.

Similarly we have

$$A^4 R_{D_+}(A,h) - A^{-4} R_{D_-}(A,h) + \left(A^2 - A^{-2}\right) R_{D_0}(A,h) = 0$$

for the Miyazawa polynomials. Some relations are given in [2,5] for the Jones–Kauffman polynomials of links in a virtual skein triple (D_+, D_-, D_v). In this paper, we give some relations for the Miyazawa polynomials of links in a virtual skein triple (Theorems 2 and 4), which are generalization of the relations given in [2,5].

2. Definitions and results

A *magnetic graph* in the 3-sphere S^3 is a 2-valent graph G in S^3 such that the edges of G are oriented alternately as in Fig. 3. Throughout this paper, we allow graphs to have circular components consisting of closed edges without vertices.

A *magnetic graph diagram* is a projection image of a magnetic graph on a plane equipped with over/under information on each crossing. See Fig. 4, for example.

A *virtual magnetic graph diagram*, which is written as VMG diagram for short, is a magnetic graph diagram possibly with some virtual crossings. See Fig. 5. If two VMG diagrams are related by a finite sequence of generalized Reidemeister moves, they are said to be *equivalent*. Virtual link diagrams are VMG diagrams without vertices.

Let D be a VMG diagram whose crossings are all virtual and $e(D)$ the set of edges of D. A *weight map* of D is a map $\sigma : e(D) \to \{+1, -1\}$ such that $\sigma(e) \neq \sigma(e')$ for adjacent edges e and e' of D. See Fig. 6, for example.

Fig. 3.

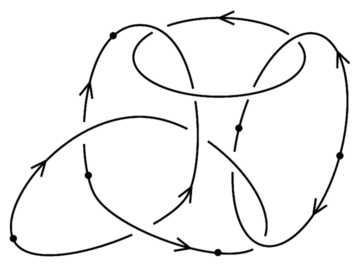

Fig. 4.

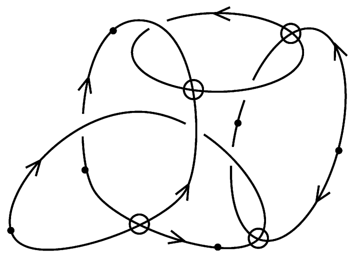

Fig. 5.

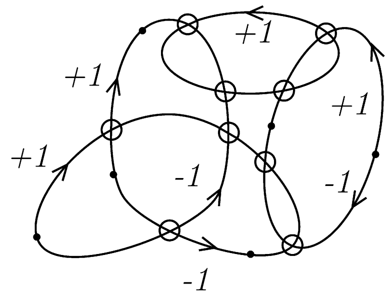

Fig. 6.

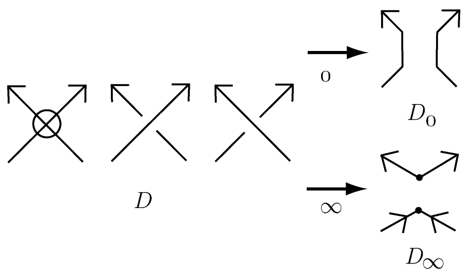

Fig. 7.

When a weight map σ is given, for a virtual crossing v of D where two edges e and e' (possibly $e = e'$) intersect, the *parity* of v with respect to σ is defined to be the product of $\sigma(e)$ and $\sigma(e')$ and denoted by $\mathrm{i}_\sigma(v)$. We call v a *regular crossing* or an *irregular crossing* with respect to σ according as $\mathrm{i}_\sigma(v) = +1$ or $\mathrm{i}_\sigma(v) = -1$, respectively. The *parity* of D is defined to be the product of parities over all virtual crossings. Lemma 1 of [4] shows that the parity of a VMG diagram D does not depend on a weight map σ. Thus, we denote it by $\mathrm{i}(D)$. Note that $\mathrm{i}(D) = +1$ if the number of irregular crossings of D (with respect to any weight map σ) is even; otherwise $\mathrm{i}(D) = -1$.

For example, there are 5 irregular crossings in the VMG diagram illustrated in Fig. 6, and the parity of this diagram is -1.

Let D be a VMG diagram, and let p be a (real or virtual) crossing. By 0-*splice* (or ∞-*splice*) at p, we mean a local replacement illustrated in Fig. 7.

A *state* of D is a VMG diagram obtained from D by applying 0-splice or ∞-splice at each real crossing, and virtual crossings of D are intact. Our states correspond to *oriented states* in [6]. For a state S of D, we denote by $C_0(D; S)$ (or $C_\infty(D; S)$) the set of real crossings of D where 0-splices (or ∞-splices) are applied to obtain the state S from D. We also denote the sign of a real crossing p by $\mathrm{sign}(p)$.

For a VMG diagram D, we define

$$H_D(A,h) = \sum_S A^{\natural(S,D)}\left(-A^2 - A^{-2}\right)^{\sharp S - 1} h^{(1-\mathrm{i}(S))/2} \in \mathbf{Z}\left[A, A^{-1}, h\right],$$

where S runs over all states of D and $\natural(S, D)$ is $\sum_{p \in C_0(D;S)} \mathrm{sign}(p) - \sum_{p \in C_\infty(D;S)} \mathrm{sign}(p)$, $\sharp S$ is the number of components of S, and $\mathrm{i}(S)$ is the parity of S. We define

$$R_D(A,h) = \left(-A^3\right)^{-\omega(D)} H_D(A,h),$$

where $\omega(D)$, called the *writhe*, is the number of positive crossings minus that of negative crossings of D. Then $R_D(A,h)$ is preserved under generalized Reidemeister moves [4]. We call $R_D(A,h)$ the *Miyazawa polynomial* of D.

Note that, for a VMG diagram D, exponents of h in the terms of $R_D(A,h)$ are 0 and 1. Thus $R_D(A,h)$ can be written as $\Phi_D(A) + \Psi_D(A)h$, where $\Phi_D(A)$ and $\Psi_D(A)$ are elements of $\mathbf{Z}[A, A^{-1}]$. Then $\Phi_D(A)$ and $\Psi_D(A)$ are invariants of D under generalized Reidemeister moves.

Remark. When D is a virtual link diagram, by the definition of the bracket polynomial $\langle D\rangle$ and the Jones–Kauffman polynomial $f_D(A)$ in [7], we have $\langle D\rangle = H_D(A,1)$ and $f_D(A) = R_D(A,1)$. In particular, $f_D(A) = \Phi_D(A) + \Psi_D(A)$.

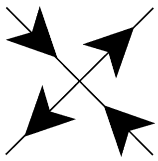

Fig. 8.

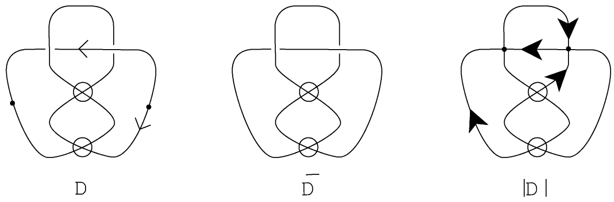

Fig. 9.

Theorem 1. [4]. *For a μ-component virtual link diagram D, we have $\Phi_D(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2(\mu-1)}$ and $\Psi_D(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2\mu}$.*

Let D be a VMG diagram. The unoriented virtual link diagram obtained from D by forgetting all vertices and orientations of all edges of D, is denoted by $\bar{D}$. We denote the 4-valent graph, which is obtained from $\bar{D}$ by replacing all real crossings of D with vertices, by $|D|$. We say that D or $|D|$ admits an *alternate orientation* if all of edges of $|D|$ admits an orientation as in Fig. 8 at each vertex. A VMG diagram D depicted in Fig. 9 admits an alternate orientation.

Theorem 2. *Let (D_+, D_-, D_v) be a virtual skein triple. Suppose that D_+ admits an alternate orientation. Then*

$$A^3\left(A^3 - A^{-3}h\right)R_{D_+}(A,h) + A^{-3}\left(A^3h - A^{-3}\right)R_{D_-}(A,h) = \left(A^6 - A^{-6}\right)R_{D_v}(A,h).$$

We prove Theorem 2 in Section 4.

Throughout this paper for a virtual skein triple (D_+, D_-, D_v) with respect to a crossing p_+ of D_+, suppose that p_+ is illustrated in Fig. 2 and p_- (or p_v) is the negative crossing (or the virtual crossing) of D_- (or D_v) which is replaced with p_+ illustrated in Fig. 2.

A virtual link diagram D is said to be a virtual link diagram of *type 0* (or *type* ∞ respectively) *with respect to* a real crossing p, if the virtual link diagram obtained from D by applying 0-splice (respectively ∞-splice) at p admits an alternate orientation. Then we have the following.

Proposition 3. (Cf. [2, Proposition 4].) *Let (D_+, D_-, D_v) be a virtual skein triple with respect to a positive crossing p_+ of D_+. Suppose that D_v admits an alternate orientation. Then D_+ is of type 0 or type ∞ with respect to p_+, and so is D_- with respect to the negative crossing p_-.*

Proof. Suppose that the crossing p_+ of D_+ is illustrated in Fig. 10(a). Since $|D_v|$ admits an alternate orientation, we may assume that the alternate orientation is as in Fig. 10(b) or (d). Let D_0 or D_∞ be a VMG diagram obtained from D_+ by applying 0-splice or ∞-splice at p_+, respectively. If $|D_v|$ admits an alternate orientation as in Fig. 10(b) (or Fig. 10 (d), respectively), then the 4-valent graph $|D_0|$ (respectively $|D_\infty|$) admits an alternate orientation induced from $|D_v|$ as in (c) (respectively (e)) of Fig. 10. Then D_+ is of type 0 (respectively type ∞) with respect to p_+. The case for D_- is proved similarly. □

Remark. (1) In [1], the notion of "checkerboard colorability" for virtual link diagrams is introduced. It is known that a virtual link diagram D admits an alternate orientation if and only if D admits a checkerboard coloring, [3,5].

(2) In [2], the notion of *type A* or *type B* is defined. For a virtual skein triple (D_+, D_-, D_v) with respect to a positive crossing p_+ of D_+, D_+ is of type A (or type B) with respect to p_+ and D_- is of type B (or type A) with respect to p_-, if and only if D_+ is of type 0 (or type ∞) with respect to p_+ and so is D_- with respect to p_-. Note that D_+ is of type A (or type B) with respect to p_+ if D_- is of type B (or type A) with respect to p_-. Thus, Proposition 3 in this paper is equivalent to Proposition 4 of [2].

We abbreviate $R_D(A,h)$ to R_D. We prove the following theorem in Section 4.

Theorem 4. *Let (D_+, D_-, D_v) be a virtual skein triple with respect to a positive crossing p_+ of D_+. Suppose that D_v admits an alternate orientation.*

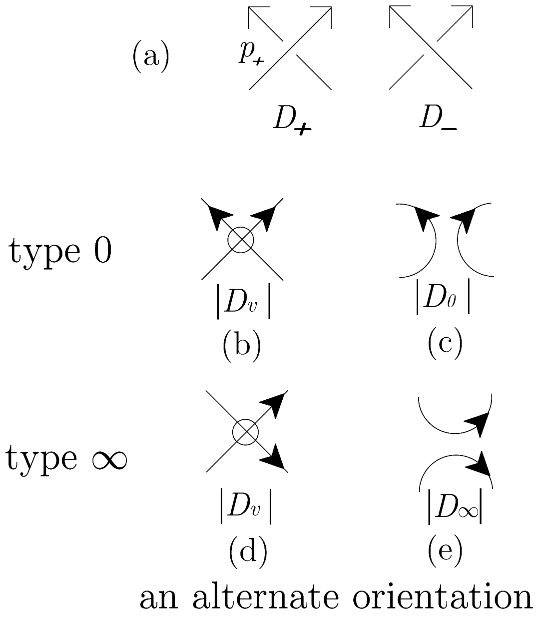

Fig. 10.

(1) *If D_+ is of type* 0 *with respect to p_+, then*

$$A(A^3h - A^{-3})R_{D_+} + A^{-1}(A^3 - A^{-3}h)R_{D_-} = (A^2 - A^{-2})(1 + (A^2 + A^{-2})h)R_{D_v}.$$

(2) *If D_+ is of type ∞ with respect to p_+, then*

$$A^5(A^3h - A^{-3})R_{D_+} + A^{-5}(A^3 - A^{-3}h)R_{D_-} = (A^2 - A^{-2})(h + A^2 + A^{-2})hR_{D_v}.$$

Let $R_{D_+} = \Phi_+ + \Psi_+h$, $R_{D_-} = \Phi_- + \Psi_-h$ and $R_{D_v} = \Phi_v + \Psi_vh$, where Φ_+, Ψ_+, Φ_-, Ψ_-, Φ_v and Ψ_v are elements of $\mathbf{Z}[A, A^{-1}]$.

Theorem 5. *Under the same assumption as Theorem* 4*, we have*

$$R_{D_v} = \Phi_+ + A^6\Psi_+ = \Phi_- + A^{-6}\Psi_-.$$

Proof. The equation in the case (1) of Theorem 4 is written as

$$\begin{aligned}&(A^4\Psi_+ - A^{-4}\Psi_-)h^2 + (A^4\Phi_+ - A^{-4}\Phi_- - A^{-2}\Psi_+ + A^2\Psi_-)h - A^{-2}\Phi_+ + A^2\Phi_- \\ &\quad = A^{-2}(A^4 - 1)R_{D_v} + A^{-4}(A^4 - 1)(A^4 + 1)R_{D_v}h.\end{aligned}$$

Since D_v admits an alternate orientation [4, Theorem 16] R_{D_v} is valued in $\mathbf{Z}[A, A^{-1}]$. Thus we have

$$\begin{aligned}&A^4\Psi_+ - A^{-4}\Psi_- = 0, \\ &A^{-2}(A^4 - 1)R_{D_v} = -A^{-2}\Phi_+ + A^2\Phi_-\end{aligned}$$

and

$$A^{-4}(A^4 - 1)(A^4 + 1)R_{D_v} = A^4\Phi_+ - A^{-4}\Phi_- - A^{-2}\Psi_+ + A^2\Psi_-.$$

Hence we have the conclusion.

In the case (2) of Theorem 4 we have the conclusion similarly. □

By Theorem 5, we have the following.

Corollary 6. *Let (D_+, D_-, D_v) be a virtual skein triple such that D_+ and D_- are classical link diagram. If the Jones–Kauffman polynomial of D_+ is different from that of D_-, then D_v admits no alternate orientation.*

3. Parities of state

Let D be a virtual link diagram and S a state of D. A real crossing p of D is said to be *normal with respect to S* if $\sharp S \neq \sharp S'$, where S' is the state obtained from S by changing the splice at p, [3,5].

Lemma 7. [3,5] *Let D be a virtual link diagram which admits an alternate orientation. Then each real crossing of D is normal with respect to any state of D.*

Proof. Let S be a state of D, and let p be a crossing of D. Assume that $|D|$ admits an alternate orientation near p as in Fig. 11(a). Then $\bar{S}$ admits an orientation induced from an alternate orientation of D as illustrated in Fig. 11(b) or (c).

When $\bar{S}$ admits the orientation illustrated in (b) (or (c), respectively) of Fig. 11, the loops passing nearby p in $\bar{S}$ are illustrated in Fig. 11(d) or (f) (respectively Fig. 11(e) or (g)). A state in Fig. 11(d) or (f) is obtained from a state in Fig. 11(e) or (g) by changing the splice at the crossing, respectively. Hence the number of components changes. $\square$

Lemma 8. [4, Lemma 14] *Let D be a virtual link diagram with a real crossing p. Let S be a state of D and S' the state of D obtained from S by changing the splice at p.*

(1) *If p is normal with respect to S, then* $\mathrm{i(S)} = \mathrm{i(S')}$.
(2) *If p is not normal with respect to S, then* $\mathrm{i(S)} = -\mathrm{i(S')}$.

Lemma 9. (Cf. [4, Theorem 16].) *Let D be a virtual link diagram which admits an alternate orientation. Then for any state S of D, we have* $\mathrm{i(S)} = 1$, *which means that the number of irregular crossings of S is even with respect to any weight map of S.*

Proof. Let S_0 be the state obtained from D by applying 0-splices at all real crossings. Then $\mathrm{i}(S_0) = 1$ since S_0 consists of closed edges without vertices and S_0 has a weight map sending all edges to 1. A state S of D is obtained from S_0 by changing the splices at some real crossings. Since D admits an alternate orientation, any real crossing of D is normal with respect to any state of D by Lemma 7. Applying Lemma 8 inductively, we have $\mathrm{i}(S) = 1$. $\square$

Let (D_+, D_-, D_v) be a virtual skein triple with respect to a crossing p_+ of D_+. A state of D_+ is any of six kinds of states listed in Fig. 12 according to how to apply splice at p_+ and how to connect arcs outside $N(p_+)$. Let STA_1^0, STA_2^0 and STA_3^0 be the sets of states of D_+ with 0-splice at p_+ as illustrated in Fig. 12(a), (d) and (g), respectively. Let STA_1^∞, STA_2^∞ and STA_3^∞ be the sets of states of D_+ with ∞-splice at p_+ as illustrated in Fig. 12(b), (e) and (h),

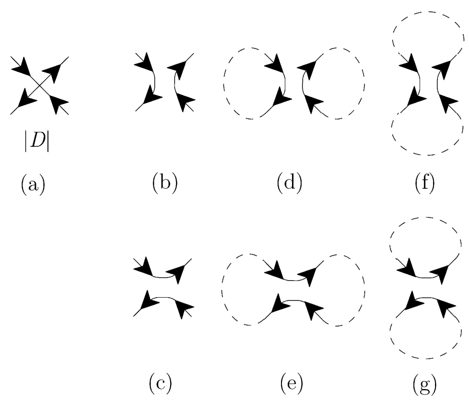

Fig. 11.

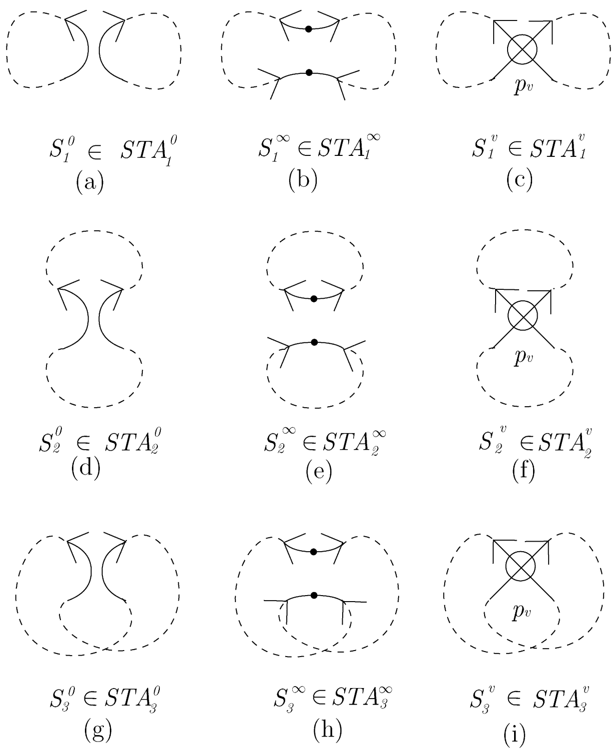

Fig. 12.

respectively. The set of states of D_+ consists of these six sets which are disjoint. Since D_- is obtained from D_+ by replacing p_+ with a negative crossing, $|D_+|$ and $|D_-|$ are identical. Hence the sets of states of D_+ are regarded as the sets of states of D_-. As for the states of D_v , we have three sets STA_1^v, STA_2^v and STA_3^v whose elements are as illustrated in Fig. 12(c), (f) and (i) respectively. Note that these three are disjoint. Since D_v is obtained from D_+ by replacing a positive crossing p_+ with a virtual crossing p_v, $D_v \setminus N(p_v)$ is identical to $D_+ \setminus N(p_+)$. There is a one-to-one correspondence, say η_0 (or η_∞), from the set of states of D_v to the set of states of D_+ with 0-splice (or ∞-splice) at p_+, such that for a state, S of D_v, $\eta_0(S)$ (or $\eta_\infty(S)$) is obtained from S by applying 0-splice (or ∞-splice) at the virtual crossing p_v. Then we have the following.

Claim 10. *Let S be a state of D_v. If S belongs to STA_j^v, then $\eta_0(S)$ belongs to STA_j^0 and $\eta_\infty(S)$ to STA_j^∞ for $j = 1, 2, 3$.*

Throughout Sections 3 and 4, for a virtual skein triple (D_+, D_-, D_v) with respect to a crossing p_+ of D_+, suppose that S_j^v is a state of D_v in STA_j^v and S_j^0 (or S_j^∞) is the state of D_+ which is $\eta_0(S_j^v)$ in STA_j^0 (or $\eta_\infty(S_j^v)$ in STA_j^∞), where $j = 1, 2, 3$.

Lemma 11. *Let (D_+, D_-, D_v) be a virtual skein triple with respect to a positive crossing p_+ of D_+. If D_+ admits an alternate orientation, then STA_3^0, STA_3^∞ and STA_3^v are the empty set.*

Proof. As in the proof of Lemma 7, since D_+ admits an alternate orientation, a state of D_+ with 0-splice (or ∞-splice, respectively) at p_+ belongs to either STA_1^0 or STA_2^0 (or either STA_1^∞ or STA_2^∞). Since the sets of states of D_+ are regarded as the sets of states of D_-, so is a state of D_-. By Claim 10 any state of D_v belongs to either STA_1^v or STA_2^v. $\square$

Lemma 12. *Let (D_+, D_-, D_v) be a virtual skein triple with respect to a crossing p_+ of D_+. Suppose that D_v admits an alternate orientation.*

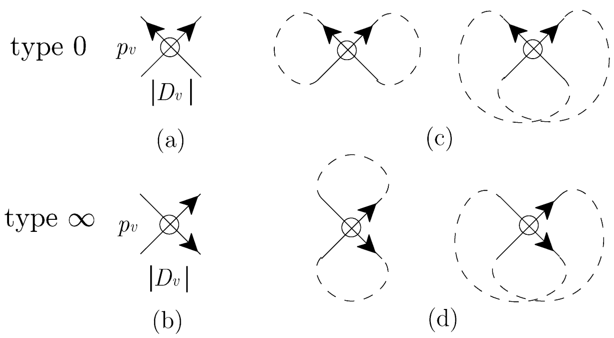

Fig. 13.

(1) *If D_+ is of type* 0 *with respect to p_+, then* STA_2^0, STA_2^∞ *and* STA_2^v *are the empty set.*
(2) *If D_+ is of type ∞ with respect to p_+, then* STA_1^0, STA_1^∞ *and* STA_1^v *are the empty set.*

Proof. Suppose that D_+ is of type 0 (or type ∞, respectively) with respect to p_+. Then $|D_v|$ admits an alternate orientation as in (a) (respectively (b)) of Fig. 13.

Any state of D_v admits an orientation induced from an alternate orientation of $|D_v|$ as stated in the proof of Lemma 7. (See Fig. 11(a), (b) and (c).) So any state of D_v is either of two kinds of states illustrated in Fig. 13(c) (or Fig. 13(d), respectively). Then any state of D_v belongs to either STA_1^v or STA_3^v (respectively STA_2^v or STA_3^v). By Claim 10 any state of D_+ with 0-splice at p_+ belongs to either STA_1^0 or STA_3^0 (respectively STA_2^0 or STA_3^0) and any state of D_+ with ∞-splice at p_+ to either STA_1^∞ or STA_3^∞ (respectively STA_2^∞ or STA_3^∞). □

Lemma 13. *Let (D_+, D_-, D_v) be a virtual skein triple with respect to a crossing p_+ of D_+ and S a state of D_v. Suppose that D_+ admits an alternate orientation. If S belongs to* STA_1^v (*or* STA_2^v)*, then* $\mathrm{i}(S) = 1$ (*or* $\mathrm{i}(S) = -1$).

Proof. Note that, for $j = 1, 2$, $S_j^0 \setminus N(p_+) = S_j^\infty \setminus N(p_+) = S_j^v \setminus N(p_v)$, denoted by G_j, consists of two arcs and some loops. There are an even (or odd, respectively) number of vertices on each of two arcs in G_1 (respectively G_2). Hence the virtual crossing p_v is regular in S_1^v (respectively irregular in S_2^v) with respect to any weight map of S_1^v (respectively S_2^v). A weight map σ_1^v of S_1^v (respectively σ_2^v of S_2^v) induces a weight map σ_1^0 of S_1^0 (respectively σ_2^∞ of S_2^∞) such that $\sigma_1^0(e) = \sigma_1^v(e)$ (respectively $\sigma_2^\infty(e) = \sigma_2^v(e)$) for an edge e of G_1 (respectively G_2). So the number of irregular crossings of S_1^v (respectively S_2^v) with respect to σ_1^v (respectively σ_2^v) is congruent to that of S_1^0 with respect to σ_1^0 (respectively is not congruent to that of S_2^∞ with respect to σ_2^∞) modulo 2. Since D_+ admits an alternate orientation, the number of irregular crossings of S_1^0 (respectively S_2^∞) is even with respect to any weight map of S_1^0 (respectively S_2^∞) by Lemma 9. Thus we see that $\mathrm{i}(S_1^v) = 1$ (respectively $\mathrm{i}(S_2^v) = -1$), since the parity of a VMG diagram does not depend on a weight map. □

Lemma 14. *Let (D_+, D_-, D_v) be a virtual skein triple with respect to a crossing p_+ of D_+. Suppose that D_v admits an alternate orientation and D_+ is of type* 0 *with respect to p_+.*

(1) *If S is a state in* $\mathrm{STA}_1^0 \cup \mathrm{STA}_1^\infty$*, then* $\mathrm{i}(S) = 1$.
(2) *If S is a state in* STA_3^0 (*or* STA_3^∞)*, then* $\mathrm{i}(S) = 1$ (*or* $\mathrm{i}(S) = -1$).

Proof. Note that, for $j = 1, 3$, $S_j^0 \setminus N(p_+) = S_j^\infty \setminus N(p_+) = S_j^v \setminus N(p_v)$, denoted by G_j, consists two arcs and some loops. There are an even number of vertices on each of two arcs of G_1. Thus the virtual crossing p_v is regular in S_1^v with respect to any weight map of S_1^v. A weight map σ_1^v of S_1^v induces a weight map σ_1^0 of S_1^0 such that $\sigma_1^0(e) = \sigma_1^v(e)$ for an edge $e \in G_1$. So the number of irregular crossings of S_1^0 with respect to σ_1^0 is equal to that of S_1^v with respect to σ_1^v. Since D_v admits an alternate orientation, we have $\mathrm{i}(S_1^v) = 1$ by Lemma 9, which implies $\mathrm{i}(S_1^0) = 1$. The real crossing p_+ of D_+ is normal with respect to S_1^0. Hence we have $\mathrm{i}(S_1^\infty) = 1$ by Lemma 8.

A weight map σ_3^v of S_3^v such that the virtual crossing p_v is regular with respect to σ_3^v, induces a weight map σ_3^0 of S_3^0 such that $\sigma_3^0(e) = \sigma_3^v(e)$ for an edge $e \in G_3$. Then we see that the number of irregular crossings of S_3^0 with respect to σ_3^0 is equal to that of S_3^v with respect to σ_3^v. Since D_v admits an alternate orientation, we have $\mathrm{i}(S_3^v) = 1$. Hence, we have $\mathrm{i}(S_3^0) = 1$, which implies $\mathrm{i}(S_3^\infty) = -1$ by Lemma 8 since the real crossing p_+ of D_+ is not normal with respect to S_3^0. □

Lemma 15. *Let (D_+, D_-, D_v) be a virtual skein triple with respect to a crossing p_+ of D_+. Suppose that D_v admits an alternate orientation and D_+ is of type ∞ with respect to p_+.*

(1) *If S is a state in $\mathrm{STA}_2^0 \cup \mathrm{STA}_2^\infty$, then* $\mathrm{i}(S) = -1$.
(2) *If S is a state in STA_3^0 (or STA_3^∞), then* $\mathrm{i}(S) = 1$ (*or* $\mathrm{i}(S) = -1$).

Proof. Note that $S_2^0 \setminus N(p_+) = S_2^\infty \setminus N(p_+) = S_2^v \setminus N(p_v)$, denoted by G_2, consists two arcs and some loops. There are an odd number of vertices on each of two arcs in G_2. Then the virtual crossing p_v is irregular in S_2^v with respect to any weight map of S_2^v. A weight map σ_2^v of S_2^v induces a weight map σ_2^∞ of S_2^∞ such that $\sigma_2^\infty(e) = \sigma_2^v(e)$ for an edge $e \in G_2$. So the number of irregular crossings of S_2^∞ with respect to σ_2^∞ is not congruent to that of S_2^v with respect to σ_2^v modulo 2. Since D_v admits an alternate orientation, the number of irregular crossings of S_2^v with respect to any weight map of S_2^v is even by Lemma 9. Hence we have $\mathrm{i}(S_2^\infty) = -1$, which implies $\mathrm{i}(S_2^0) = -1$ by Lemma 8 since the real crossing p_+ of D_+ is normal with respect to S_2^0. We have $\mathrm{i}(S_3^0) = 1$ and $\mathrm{i}(S_3^\infty) = -1$ by an argument similar to the case where D_+ is of type 0. □

4. Proofs of Theorems 2 and 4

Lemma 16. *For a virtual skein triple (D_+, D_-, D_v) with respect to a positive crossing p_+ of D_+, we have*

$$\sum_{S_1^0 \in \mathrm{STA}_1^0} A^{\natural(S_1^0, D_\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_1^0 - 1} = \sum_{S_1^v \in \mathrm{STA}_1^v} A^{\natural(S_1^v, D_v) + \mathrm{b}(\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_1^v},$$

$$\sum_{S_1^\infty \in \mathrm{STA}_1^\infty} A^{\natural(S_1^\infty, D_\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_1^\infty - 1} = \sum_{S_1^v \in \mathrm{STA}_1^v} A^{\natural(S_1^v, D_v) - \mathrm{b}(\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_1^v - 1},$$

$$\sum_{S_2^0 \in \mathrm{STA}_2^0} A^{\natural(S_2^0, D_\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^0 - 1} = \sum_{S_2^v \in \mathrm{STA}_2^v} A^{\natural(S_2^v, D_v) + \mathrm{b}(\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^v - 1},$$

$$\sum_{S_2^\infty \in \mathrm{STA}_2^\infty} A^{\natural(S_2^\infty, D_\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^\infty - 1} = \sum_{S_2^v \in \mathrm{STA}_2^v} A^{\natural(S_2^v, D_v) - \mathrm{b}(\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^v},$$

$$\sum_{S_3^0 \in \mathrm{STA}_3^0} A^{\natural(S_3^0, D_\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^0 - 1} = \sum_{S_3^v \in \mathrm{STA}_3^v} A^{\natural(S_3^v, D_v) + \mathrm{b}(\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^v - 2}$$

and

$$\sum_{S_3^\infty \in \mathrm{STA}_3^\infty} A^{\natural(S_3^\infty, D_\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^\infty - 1} = \sum_{S_3^v \in \mathrm{STA}_3^v} A^{\natural(S_3^v, D_v) - \mathrm{b}(\varepsilon)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^v - 2},$$

where ε is $+$ or $-$, and if $\varepsilon = +$, then $b(\varepsilon)$ is 1; *otherwise* $\mathrm{b}(\varepsilon) = -1$.

Proof. We see that $\sharp S_1^v = \sharp S_1^0 - 1 = \sharp S_1^\infty$, $\sharp S_2^v = \sharp S_2^0 = \sharp S_2^\infty - 1$ and $\sharp S_3^v = \sharp S_3^0 + 1 = \sharp S_3^\infty + 1$. For $j = 1, 2, 3$, $\natural(S_j^v, D_v) = \natural(S_j^0, D_+) - 1 = \natural(S_j^\infty, D_+) + 1$ and $\natural(S_j^v, D_v) = \natural(S_j^0, D_-) + 1 = \natural(S_j^\infty, D_-) - 1$ hold. Thus we have the conclusion. □

Proof of Theorem 2. By Lemmas 9 and 11 we have

$$H_{D_+} = \sum_{S_1^0 \in \mathrm{STA}_1^0} A^{\natural(S_1^0, D_+)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^0 - 1} + \sum_{S_2^0 \in \mathrm{STA}_2^0} A^{\natural(S_2^0, D_+)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_2^0 - 1}$$
$$+ \sum_{S_1^\infty \in \mathrm{STA}_1^\infty} A^{\natural(S_1^\infty, D_+)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^\infty - 1} + \sum_{S_2^\infty \in \mathrm{STA}_2^\infty} A^{\natural(S_2^\infty, D_+)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_2^\infty - 1}$$

and

$$H_{D_-} = \sum_{S_1^0 \in \mathrm{STA}_1^0} A^{\natural(S_1^0, D_-)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^0 - 1} + \sum_{S_2^0 \in \mathrm{STA}_2^0} A^{\natural(S_2^0, D_-)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_2^0 - 1}$$
$$+ \sum_{S_1^\infty \in \mathrm{STA}_1^\infty} A^{\natural(S_1^\infty, D_-)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^\infty - 1} + \sum_{S_2^\infty \in \mathrm{STA}_2^\infty} A^{\natural(S_2^\infty, D_-)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_2^\infty - 1}.$$

By Lemmas 11 and 13 we have

$$H_{D_v} = \sum_{S_1^v \in \mathrm{STA}_1^v} A^{\natural(S_1^v, D_v)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^v - 1} + \sum_{S_2^v \in \mathrm{STA}_2^v} A^{\natural(S_2^v, D_v)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_2^v - 1} \cdot h.$$

Then, by Lemma 16 we have

$$H_{D_+} = -A^3 \sum_{S_1^v \in \mathrm{STA}_1^v} A^{\natural(S_1^v, D_v)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^v - 1} - A^{-3} \sum_{S_2^v \in \mathrm{STA}_2^v} A^{\natural(S_2^v, D_v)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_2^v - 1}$$

and

$$H_{D_-} = -A^{-3} \sum_{S_1^v \in \mathrm{STA}_1^v} A^{\natural(S_1^v, D_v)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^v - 1} - A^{3} \sum_{S_2^v \in \mathrm{STA}_2^v} A^{\natural(S_2^v, D_v)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_2^v - 1}.$$

So we have

$$\bigl(A^6 - A^{-6}\bigr) H_{D_v} = -\bigl(A^3 - A^{-3}h\bigr) H_{D_+} - \bigl(A^3 h - A^{-3}\bigr) H_{D_-}.$$

Hence we have the conclusion, since $H_{D_+} = (-A^3)^{\omega(D_+)} R_{D_+} = (-A^3)^{\omega(D_v)+1} R_{D_+}$, $H_{D_-} = (-A^3)^{\omega(D_-)} R_{D_+} = (-A^3)^{\omega(D_v)-1} R_{D_-}$ and $H_{D_v} = (-A^3)^{\omega(D_v)} R_{D_v}$. □

Proof of Theorem 4. Suppose that D_+ is of type 0 with respect to p_+. Since D_v admits an alternate orientation, the parity of any state of D_v is $+1$ by Lemma 9. So, by Lemma 12(1) we have

$$H_{D_v} = \sum_{S_1^v \in \mathrm{STA}_1^v} A^{\natural(S_1^v, D_v)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^v - 1} + \sum_{S_3^v \in \mathrm{STA}_3^v} A^{\natural(S_3^v, D_v)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_3^v - 1}.$$

By Lemmas 12(1) and 14 we have

$$H_{D_+} = \sum_{S_1^0 \in \mathrm{STA}_1^0} A^{\natural(S_1^0, D_+)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^0 - 1} + \sum_{S_3^0 \in \mathrm{STA}_3^0} A^{\natural(S_3^0, D_+)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_3^0 - 1}$$
$$+ \sum_{S_1^\infty \in \mathrm{STA}_1^\infty} A^{\natural(S_1^\infty, D_+)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^\infty - 1} + \sum_{S_3^\infty \in \mathrm{STA}_3^\infty} A^{\natural(S_3^\infty, D_+)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_3^\infty - 1} \cdot h.$$

Similarly we have

$$H_{D_-} = \sum_{S_1^0 \in \mathrm{STA}_1^0} A^{\natural(S_1^0, D_-)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^0 - 1} + \sum_{S_3^0 \in \mathrm{STA}_3^0} A^{\natural(S_3^0, D_-)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_3^0 - 1}$$
$$+ \sum_{S_1^\infty \in \mathrm{STA}_1^\infty} A^{\natural(S_1^\infty, D_-)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_1^\infty - 1} + \sum_{S_3^\infty \in \mathrm{STA}_3^\infty} A^{\natural(S_3^\infty, D_-)} \bigl(-A^2 - A^{-2}\bigr)^{\sharp S_3^\infty - 1} \cdot h.$$

Then, by Lemma 16 we have

$$H_{D_+} = -A^3 \sum_{S_1^v \in \mathrm{STA}_1^v} A^{\natural(S_1^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_1^v - 1} + \left(A + A^{-1}h\right) \sum_{S_3^v \in \mathrm{STA}_3^v} A^{\natural(S_3^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^v - 2}$$

and

$$H_{D_-} = -A^{-3} \sum_{S_1^v \in \mathrm{STA}_1^v} A^{\natural(S_1^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_1^v - 1} + \left(A^{-1} + Ah\right) \sum_{S_3^v \in \mathrm{STA}_3^v} A^{\natural(S_3^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^v - 2}.$$

So we have

$$\left(A^2 - A^{-2}\right)\left(1 + \left(A^2 + A^{-2}\right)h\right) H_{D_v} = A^{-2}\left(A^{-3} - A^3 h\right) H_{D_+} + A^2\left(A^{-3}h - A^3\right) H_{D_-}.$$

Hence we have the conclusion, since $H_{D_+} = (-A^3)^{\omega(D_+)} R_{D_+} = (-A^3)^{\omega(D_v)+1} R_{D_+}$, $H_{D_-} = (-A^3)^{\omega(D_-)} R_{D_-} = (-A^3)^{\omega(D_v)-1} R_{D_-}$ and $H_{D_v} = (-A^3)^{\omega(D_v)} R_{D_v}$.

Suppose that D_+ is of type ∞ with respect to the positive crossing p_+. By Lemmas 12(2) and 9 we have

$$H_{D_v}(A, h) = \sum_{S_2^v \in \mathrm{STA}_2^v} A^{\natural(S_2^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^v - 1} + \sum_{S_3^v \in \mathrm{STA}_3^v} A^{\natural(S_3^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^v - 1}.$$

By Lemmas 12(2) and 15 we have

$$\begin{aligned} H_{D_+} = {} & \sum_{S_2^0 \in \mathrm{STA}_2^0} A^{\natural(S_2^0, D_+)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^0 - 1} \cdot h + \sum_{S_3^0 \in \mathrm{STA}_3^0} A^{\natural(S_3^0, D_+)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^0 - 1} \\ & + \sum_{S_2^\infty \in \mathrm{STA}_2^\infty} A^{\natural(S_2^\infty, D_+)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^\infty - 1} \cdot h + \sum_{S_3^\infty \in \mathrm{STA}_3^\infty} A^{\natural(S_3^\infty, D_+)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^\infty - 1} \cdot h. \end{aligned}$$

Similarly we have

$$\begin{aligned} H_{D_-} = {} & \sum_{S_2^0 \in \mathrm{STA}_2^0} A^{\natural(S_2^0, D_-)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^0 - 1} \cdot h + \sum_{S_3^0 \in \mathrm{STA}_3^0} A^{\natural(S_3^0, D_-)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^0 - 1} \\ & + \sum_{S_2^\infty \in \mathrm{STA}_2^\infty} A^{\natural(S_2^\infty, D_-)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^\infty - 1} \cdot h + \sum_{S_3^\infty \in \mathrm{STA}_3^\infty} A^{\natural(S_3^\infty, D_-)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^\infty - 1} \cdot h. \end{aligned}$$

Thus, by Lemma 16 we obtain

$$H_{D_+} = -A^{-3}h \sum_{S_2^v \in \mathrm{STA}_2^v} A^{\natural(S_2^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^v - 1} + \left(A + A^{-1}h\right) \sum_{S_3^v \in \mathrm{STA}_3^v} A^{\natural(S_3^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^v - 2}$$

and

$$H_{D_-} = -A^{3}h \sum_{S_2^v \in \mathrm{STA}_2^v} A^{\natural(S_2^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_2^v - 1} + \left(A^{-1} + Ah\right) \sum_{S_3^v \in \mathrm{STA}_3^v} A^{\natural(S_3^v, D_v)} \left(-A^2 - A^{-2}\right)^{\sharp S_3^v - 2}.$$

So we have

$$\left(A^2 - A^{-2}\right)\left(A^2 + A^{-2} + h\right) h H_{D_v} = A^{2}\left(A^{-3} - A^3 h\right) H_{D_+} + A^{-2}\left(A^{-3}h - A^3\right) H_{D_-}.$$

Hence we have the conclusion, since $H_{D_+} = (-A^3)^{\omega(D_+)} R_{D_+} = (-A^3)^{\omega(D_v)+1} R_{D_+}$, $H_{D_-} = (-A^3)^{\omega(D_-)} R_{D_+} = (-A^3)^{\omega(D_v)-1} R_{D_-}$ and $H_{D_v} = (-A^3)^{\omega(D_v)} R_{D_v}$. □

5. Applications

Substituting 1 for h, we have the following theorems from Theorems 2 and 4.

Theorem 17. [5] *Let (D_+, D_-, D_v) be a virtual skein triple. Suppose that D_+ (or D_-) admits an alternate orientation. Then*

$$A^3 f_{D_+}(A) + A^{-3} f_{D_-}(A) = \left(A^3 + A^{-3}\right) f_{D_v}(A).$$

Theorem 18. [2] *Let (D_+, D_-, D_v) be a virtual skein triple with respect to a positive crossing p_+ of D_+. Suppose that D_v admits an alternate orientation.*

(1) *If D_+ is of type 0 with respect to p_+, then we have*

$$Af_{D_+}(A) + A^{-1} f_{D_-}(A) = \left(A + A^{-1}\right) f_{D_v}(A).$$

(2) *If D_+ is of type ∞ with respect to p_+, then we have*

$$A^5 f_{D_+}(A) + A^{-5} f_{D_-}(A) = \left(A + A^{-1}\right) f_{D_v}(A).$$

By Theorem 2, we have the following.

Proposition 19. *Let (D_+, D_-, D_v) and (D'_+, D'_-, D'_v) be virtual skein triples such that D_+ and D'_+ admit alternate orientations. If D_v is equivalent to D'_v, then*

$$R_{D_+} = R_{D'_+} \quad \text{and} \quad R_{D_-} = R_{D\_-}.$$

Proof. Deforming the formula in Theorem 2, we have

$$\left(A^6 - A^{-6}\right) R_{D_v} = A^6 R_{D_+} - A^{-6} R_{D_-} + (R_{D_-} - R_{D_+})h.$$

The same equality holds for (D'_+, D'_-, D'_v). On the other hand, since D_+ and D'_+ admit alternate orientations, so do D_- and D'_-, and then the Miyazawa polynomials R_{D_+}, $R_{D'_+}$, R_{D_-} and $R_{D'_-}$ are valued in $\mathbf{Z}[A, A^{-1}]$ [4, Theorem 16]. Since $R_{D_v} = R_{D'_v}$, we see the equalities,

$$R_{D_-} - R_{D_+} = R_{D'_-} - R_{D'_+}$$

and

$$A^6 R_{D_+} - A^{-6} R_{D_-} = A^6 R_{D'_+} - A^{-6} R_{D'_-}.$$

Thus we have the conclusion. □

For a virtual link diagram D, we denote the mirror image of D by $D!$, which is obtained from D by switching all the real crossing of D. By Proposition 19 we have the following.

Corollary 20. *Let L be an unknotting number one knot with non-trivial Jones–Kauffman polynomial. Suppose that D is a classical knot diagram of L with a crossing point c where we apply crossing change to obtain a trivial knot diagram. Let D_v be the diagram obtained from D by replacing c with a virtual crossing. Then D_v is not equivalent to $D_v!$.*

Proof. We consider a case where c is a positive crossing. Let (D_+, D_-, D_v) be the virtual skein triple with $D = D_+$ with respect to $p_+ = c$. By assumption, D_- is a trivial knot diagram. Then $(D_-!, D_+!, D_v!)$ is a virtual skein triple. Since the Jones–Kauffman polynomial of $D = D_+$ is not trivial, that of $D_+!$ is not trivial. Since each of the diagrams D_- and $D_-!$ presents a trivial knot, the Jones–Kauffman polynomials of D_- and $D_-!$ are trivial. The

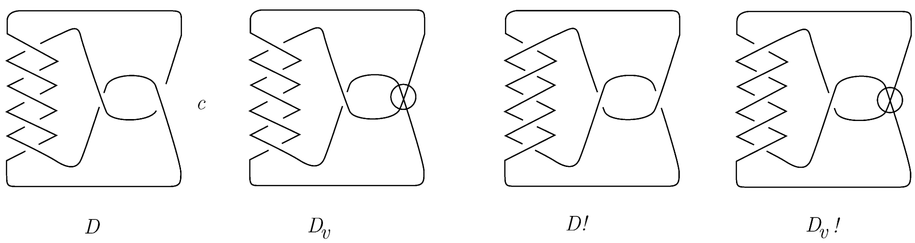

Fig. 14.

Jones–Kauffman polynomial of D_+ (or $D_-!$) is equal to R_{D_+} (or $R_{D_-!}$). Thus we have $R_{D_+} \neq R_{D_-!}$. Thus D_v is not equivalent to $D_v!$ by Proposition 19. The case where c is a negative crossing is treated similarly. □

Let D be the knot diagram illustrated in Fig. 14 whose positive crossing c where is switched to change D into a trivial knot diagram and D_v be the virtual knot diagram obtained from D by replacing c with a virtual crossing. By Corollary 20, $D_v!$ is not equivalent to D_v since the Jones–Kauffman polynomial of D is $A^{-4} - A^{-8} + 2A^{-12} - A^{-16} + 2A^{-20} - A^{-24} + A^{-28} - A^{-32}$, which is not trivial.

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